Almost sure hedging with price impact

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Joint works with G. Loeper (Monash Univ.), M. Soner (ETH Zürich), C. Zhou (NUS) and Y. Zou (ex Paris-Dauphine)

Motivation

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- BS and local (stochastic) vol models :
  - Are useful because they provide a clear hedging rule
  - Disregard frictions because do not work at high frequency
  - Taking costs into account would lead to useless degenerate prices/strategies (in theory) and is helpless. We are not working at the level of the order book.

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- However :
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  - Problematic when large positions (possibly shared) or illiquid underlying (may run after the delta)

- Question : Can we built a model which
  - Takes price impact and illiquidity into account
  - Leads to a clear hedging and pricing rule
  - Does not have embedded hidden transaction costs (otherwise the super-hedging price would be degenerate)

Some references

- Many works on hedging with illiquidity or impact : Sircar and Papanicolaou 98, Schönbucher and Wilmot 00, Frey 98, Cetin, Jarrow and Protter 04, Bank and Baum 04, Liu and Yong 05, Cetin, Soner and Touzi 09, Millot and Abergel 11, Frey and Polte 11, Almgren and Li 13, Guéant and Pu 13,...

- Illiquidity + impact + perfect hedging : Loeper 14/16 (verification arguments).

- Past and ongoing related works by D. Becherer and T. Bilarev.

Impact rule and continuous time trading dynamics

Impact rule

□ Basic rule (only permanent for the moment) : an order of δ units moves the price by

$$X_{t-} \longrightarrow X_t = X_{t-} + \delta f(X_{t-}), \quad \text{[permanent impact]}$$

and costs

$$\delta X_{t-} + \frac{1}{2}\delta^2 f(X_{t-}) = \delta \frac{X_{t-} + X_t}{2} \quad \text{[liquidity cost]}.$$

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□ We just model the curve around $\delta = 0$. This should be understood for a "small" order δ. Would obtain the same with

$$X_{t-} \longrightarrow X_t = X_{t-} + F(X_{t-}, \delta)$$

and costs

$$\int_0^\delta (X_{t-} + F(X_{t-}, \iota))d\iota$$

if $\partial_\delta F(x,0) = f(x)$, $\partial^2_{\delta x} F(x,0) = f'(x)$ and $F(x,0) = \partial^2_{\delta\delta} F(x,0) = 0$.

Trading signal and discrete trading dynamics

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□ Trade at times $t_i^n = iT/n$ (for simplicity) the quantities $\delta^n_{t_i^n} = Y_{t_i^n} - Y_{t_{i-1}^n}$.

□ We assume that the stock price evolves according to

$$X = X_{t_i^n} + \int_{t_i^n}^{\cdot} \sigma(X_s) dW_s$$

between two trades (can add a drift - or resilience effect, see Becherer and Bilarev 18).

□ The corresponding dynamics are

$$Y_t^n := \sum_{i=0}^{n-1} Y_{t_i^n} \mathbf{1}_{\{t_i^n \le t < t_{i+1}^n\}} + Y_T \mathbf{1}_{\{t=T\}} \ , \ \delta_{t_i^n}^n = Y_{t_i^n}^n - Y_{t_{i-1}^n}^n$$

$$X^n = X_0 + \int_0^{\cdot} \sigma(X_s^n) dW_s + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n),$$

$$V^n = V_0 + \int_0^{\cdot} Y_{s-}^n dX_s^n + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \frac{1}{2} (\delta_{t_i^n}^n)^2 f(X_{t_i^n -}^n),$$

where

$$V^n := \text{ cash part } + Y^n X^n = \text{“portfolio value”}.$$

□ Passing to the limit $n \to \infty$, it converges in $\mathbf{S}_2$ to

$$Y = Y_0 + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s$$

$$X = X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \underbrace{\int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} a_s(\sigma f')(X_s) ds}_{(Y_{t_i^n}^n - Y_{t_{i-1}^n}^n) f(X_{t_i^n -}^n)}$$

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \underbrace{\int_0^{\cdot} a_s^2 f(X_s) ds}_{(Y_{t_i^n}^n - Y_{t_{i-1}^n}^n)^2 f(X_{t_i^n -}^n)} ,$$

at a speed $\sqrt{n}$.

Hedging problem(s)

1. Uncovered options.
2. Covered options.
3. Covered options in a generalized model.

The case of uncovered options

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□ Super-hedging price :

$$\mathrm{v} = \inf\{\text{initial cash} \ : \ \exists (a,b) \text{ s.t. } V_T - Y_T X_T \geq g_0(X_T) \ \text{ and } \ Y_T = g_1(X_T)\}.$$

(Recall that $V = \text{cash} + YX$)

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(Recall that $V = \text{cash} + YX$)

□ Issue : needs to jump to a certain initial or final delta !

Adding jumps and splitting of large orders

- We now consider a trading signal of the form

$$Y = Y_{0-} + \int_0^{\cdot} b_s ds + \int_0^{\cdot} a_s dW_s + \int_0^{\cdot} \delta\nu(d\delta, ds)$$

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- Jumps δ_i at time τ_i is passed on $[\tau_i, \tau_i + \varepsilon]$ at a rate δ_i/ε.

□ The limit dynamics when $\varepsilon \to 0$ is

$$
\begin{aligned}
X &= X_{0-} + \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY_s^c + \int_0^\cdot a_s\sigma f'(X_s)ds \\
&\quad + \int_0^\cdot \int \Delta \mathrm{x}(X_{s-},\delta)\nu(d\delta, ds) \\
V &= V_{0-} + \int_0^\cdot Y_s dX_s^c + \frac{1}{2}\int_0^\cdot a_s^2 f(X_s)ds \\
&\quad + \int_0^\cdot \int \left(Y_{s-}\Delta \mathrm{x}(X_{s-},\delta) + \Im(X_{s-},\delta)\right)\nu(d\delta, ds).
\end{aligned}
$$

in which

$$
\Delta \mathrm{x}(x,\delta) + x = \mathrm{x}(x,\delta) := x + \int_0^\delta f(\mathrm{x}(x,s))ds
$$

$$
\text{and } \Im(x,\delta) := \int_0^\delta s f(\mathrm{x}(x,s))ds.
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Dynamic programming

- Modified geometric dynamic programming :

$$v \geq \mathrm{v}(0, x)$$

"if and only if"

$$V_\theta \geq \mathrm{v}(\theta, \mathrm{x}(X_\theta, -Y_\theta)) + \mathfrak{I}(\mathrm{x}(X_\theta, -Y_\theta), Y_\theta) \text{ for some } (a, b, \nu)$$

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- Can then apply standard stochastic target technics.

Pricing equation

- A quasi-linear pde

$$0 = -\partial_t \mathrm{v} - \hat{\mu}(\cdot, \hat{y})\partial_x[\mathrm{v} + \mathfrak{I}] - \tfrac{1}{2}\hat{\sigma}(\cdot, \hat{y})^2\partial^2_{xx}[\mathrm{v} + \mathfrak{I}]$$

where

$$\hat{\mu}(\cdot, y) := \frac{1}{2}[\partial^2_{xx}\mathrm{x}\sigma^2](\mathrm{x}(\cdot, y), -y) \text{ and } \hat{\sigma}(\cdot, y) := (\sigma\partial_x\mathrm{x})(\mathrm{x}(\cdot, y), -y),$$

and

$$\hat{y}(t, x) := \mathrm{x}^{-1}(x, x + f(x)\partial_x \mathrm{v}(t, x)).$$

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□ Terminal condition

$$G(x) := \inf\left\{y\mathrm{x}(x, y) + g_0(\mathrm{x}(x, y)) - \mathfrak{I}(x, y) : \; y = g_1(\mathrm{x}(x, y))\right\}.$$

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□ Perfect hedging : Smooth solution under additional conditions, leading to perfect hedging by following $Y = \hat{y}(\cdot, X)$.

□ For $f \equiv 0$: recovers the usual delta hedging $Y = \partial_x \mathrm{v}(\cdot, X)$.

The case of covered options

B., G. Loeper, and Y. Zou. Hedging of covered options with linear market impact and gamma constraint. *SIAM Journal on Control and Optimization*, 55(5), 3319-3348, 2017.

□ The trader receives at inception a chosen (by the trader) quantity of cash and stocks, and delivers at maturity a quantity of cash and stocks (chosen by the trader). The initial number of stocks equates the required delta to start the hedging, the quantity of stocks delivered at maturity equates the delta at maturity.

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□ Super-hedging price :

$$\mathrm{v}(t,x) := \inf\{v = c + yx \;:\; c, y, (a,b) \text{ s.t. } V_T \geq g(X_T)\}.$$

(Recall that $V = \text{cash} + YX$)

Hedging and pricing - informal derivation

Let us assume that we use the delta-hedging rule :

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and applying Itô's Lemma to $Y - \partial_x \mathrm{v}(\cdot, X)$ leads to

$$\gamma^a := \frac{a}{\sigma + fa} = \partial^2_{xx} \mathrm{v}(\cdot, X) \in \mathbb{R} \setminus \{1/f\}$$

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By definition of γ^a and a little bit of algebra :

$$\left[-\partial_t \mathrm{v} - \frac{1}{2} \frac{\sigma^2}{(1 - f \partial^2_{xx} \mathrm{v})} \partial^2_{xx} \mathrm{v} \right] (\cdot, X) = 0.$$

The pricing pde should be

$$-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v} = 0 \quad \text{on } [0, T) \times \mathbb{R},$$

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Singular pde :

- Can find smooth solutions s.t. $1 > f\partial^2_{xx}\mathrm{v}$, cf. below.
- In general, needs to take care of $1 \neq f\partial^2_{xx}\mathrm{v}$
- One possibility : add a gamma constraint $\partial^2_{xx}\mathrm{v} \leq \bar{\gamma}$ with $f\bar{\gamma} < 1$.
- A constraint of the form $f\partial^2_{xx}\mathrm{v} > 1$ does not make sense.

Hedging with a gamma contraint

- By a change of variable, we write the dynamics in the form :

$$dY = \gamma^a(X)dX + \mu_Y^{a,b}(X)dt \text{ and } dX = \sigma^a(X)dW + \mu_X^{a,b}(X)dt.$$

- We now define v with respect to the gamma constraint

$$\gamma^a(X) \leq \bar{\gamma}(X)$$

with

$$f\bar{\gamma} < 1 - \varepsilon, \quad \varepsilon > 0.$$

Pricing pde :

$$\min\left\{-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1-f\partial^2_{xx}\mathrm{v})}\partial^2_{xx}\mathrm{v}\ ,\ \bar{\gamma}-\partial^2_{xx}\mathrm{v}\right\} = 0 \quad \text{on } [0,T)\times\mathbb{R}.$$

Propagation of the gamma contraint at the boundary :

$$\mathrm{v}(T-,\cdot) = \hat{g} \quad \text{on } \mathbb{R}$$

with $\hat{g}$ the smallest (viscosity) super-solution of

$$\min\left\{\varphi - g\ ,\ \bar{\gamma}-\partial^2_{xx}\varphi\right\} = 0.$$

See Soner and Touzi 00, and Cheridito, Soner and Touzi 05.

Adding a resilience effect

- Given a speed of resilience $\rho > 0$,

$$X^n = X_0 + \int_0^{\cdot} \sigma(X_s^n) dW_s + R^n,$$

$$R^n = R_0 + \sum_{i=1}^{n} \mathbf{1}_{[t_i^n, T]} \delta_{t_i^n}^n f(X_{t_i^n -}^n) - \int_0^{\cdot} \rho R_s^n ds.$$

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- The continuous time dynamics becomes

$$X = X_0 + \int_0^{\cdot} \sigma(X_s) dW_s + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$R = R_0 + \int_0^{\cdot} f(X_s) dY_s + \int_0^{\cdot} (a_s(\sigma f')(X_s) - \rho R_s) ds$$

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2} \int_0^{\cdot} a_s^2 f(X_s) ds.$$

Extension : abstract impact model

B., G. Loeper, M. Soner and C. Zhou. Second order stochastic target problems with generalized market impact.

***Arxiv :1806.08533*, 2018.**

□ A general impact function :

$$X = x + \int_t^{\cdot} \mu(s, X_s, \gamma_s, b_s)ds + \int_t^{\cdot} \sigma(s, X_s, \gamma_s)dW_s$$

$$Y = y + \int_t^{\cdot} b_s ds + \int_t^{\cdot} \gamma_s dX_s$$

$$V = v + \int_t^{\cdot} F(s, X_s, \gamma_s)ds + \int_t^{\cdot} Y_s dX_s$$

This allows to model : permanent impact, immediate partial relaxation of the impact, modified liquidity cost, and can easily add resilience.

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This allows to model : permanent impact, immediate partial relaxation of the impact, modified liquidity cost, and can easily add resilience.

□ Relaxation of the gamma constraint. Can be as close as one wants to the singularity :

$$\min\{-\partial_t \mathrm{v} - \bar{F}(\cdot, \partial^2_{xx}\mathrm{v}) \ , \ \bar{\gamma} - \partial^2_{xx}\mathrm{v}\} = 0 \text{ on } [0, T) \times \mathbb{R},$$

where

$$\bar{F}(t, x, z) := \frac{1}{2}\sigma(t, x, z)^2 z - F(t, x, z)$$

and

$$\{\bar{F} < \infty\} = \{F < \infty\} = \{(t, x, z) : z < \bar{\gamma}(t, x)\}.$$

Expansion around 0 impact

- Scaling :

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- In the initial model, it amongs to considering ϵf in place of f.

- Expansion performed around the solution v^0 of $(\partial_z \bar{F}(\cdot, 0) =: \partial_z \bar{F}_0)$

$$\partial_t \mathrm{v}^0 + \partial_z \bar{F}_0 \partial^2_{xx} \mathrm{v}^0 = 0 \text{ on } [0, T) \times \mathbb{R} \text{ and } \mathrm{v}^0(T, \cdot) = \hat{g} \text{ on } \mathbb{R}.$$

□ **Proposition :**

$$\begin{aligned}\mathrm{v}^{\epsilon}(0,x) =& \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T [\partial_{zz}^2 \bar{F}_0 |\partial_{xx}^2 \mathrm{v}^0|^2](s,\tilde{X}_s^0)ds\right] + o(\epsilon) \\ =& \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\,\mathbb{E}\left[\partial_x \hat{g}(T,\tilde{X}_T^0)\tilde{Y}_T\right] + o(\epsilon)\end{aligned}$$

where

$$\begin{aligned}\tilde{X}^z &= x + \int_t^{\cdot} (2\partial_z \bar{F}(\cdot, z\partial_{xx}^2 \mathrm{v}^0))^{\frac{1}{2}}(s,\tilde{X}_s^z)dW_s, \\ \tilde{Y} &= \frac{1}{\sqrt{2}}\int_t^{\cdot} \frac{\partial_x\partial_z \bar{F}_0(s,\tilde{X}_s^0)\tilde{Y}_s + \partial_{zz}^2 \bar{F}_0 \partial_{xx}^2 \mathrm{v}^0(s,\tilde{X}_s^0)}{\sqrt{\partial_z \bar{F}_0(s,\tilde{X}_s^0)}}dW_s.\end{aligned}$$

□ **Proposition :**

$$\mathrm{v}^{\epsilon}(0,x) = \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\mathbb{E}\left[\int_0^T [\partial^2_{zz}\bar{F}_0|\partial^2_{xx}\mathrm{v}^0|^2](s,\tilde{X}^0_s)ds\right] + o(\epsilon)$$

$$= \mathrm{v}^0(0,x) + \frac{\epsilon}{2}\,\mathbb{E}\left[\partial_x\hat{g}(T,\tilde{X}^0_T)\tilde{Y}_T\right] + o(\epsilon)$$

where

$$\tilde{X}^z = x + \int_t^{\cdot} (2\partial_z\bar{F}(\cdot, z\partial^2_{xx}\mathrm{v}^0))^{\frac{1}{2}}(s,\tilde{X}^z_s)dW_s,$$

$$\tilde{Y} = \frac{1}{\sqrt{2}}\int_t^{\cdot} \frac{\partial_x\partial_z\bar{F}_0(s,\tilde{X}^0_s)\tilde{Y}_s + \partial^2_{zz}\bar{F}_0\partial^2_{xx}\mathrm{v}^0(s,\tilde{X}^0_s)}{\sqrt{\partial_z\bar{F}_0(s,\tilde{X}^0_s)}}dW_s.$$

□ The leading order term allows for super-hedging with $\mathbf{L}^{\infty}$-error controlled by ϵ^2.

Dual formulation

□ In the concave case :

$$\mathrm{v}(t,x) = \sup_{\mathfrak{s}} \mathbb{E}\left[\hat{g}(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}}, \mathfrak{s}_s^2)ds\right]$$
$$= \sup_{\mathfrak{s}} \mathbb{E}\left[g(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}}, \mathfrak{s}_s^2)ds\right]$$

in which

$$X^{t,x,\mathfrak{s}} = x + \int_t^{\cdot} \mathfrak{s}_s dW_s.$$

Dual formulation

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$$\mathrm{v}(t,x) = \sup_{\mathfrak{s}} \mathbb{E}\left[\hat{g}(X_T^{t,x,\mathfrak{s}}) - \int_t^T \bar{F}^*(s, X_s^{t,x,\mathfrak{s}}, \mathfrak{s}_s^2)ds\right]$$
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in which

$$X^{t,x,\mathfrak{s}} = x + \int_t^{\cdot} \mathfrak{s}_s dW_s.$$

□ In the previous model :

$$\bar{F}^*(t,x,s^2) = \frac{1}{2}\frac{(s-\sigma(t,x))^2}{f(x)}, \quad \text{for } s \geq 0.$$

Open problems

No constraint at all on the gamma ?

Dual formulation in a non-Markovian framework ?

Generic completeness ?

Existence/stability of FBSDE with impact ?

Open problems

No constraint at all on the gamma ?

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Thank you !

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